Partial Differential Equations - Resit exam

You have 3.5 hours to complete this exam. This exam is open book and open notes, but not open internet! Please show all work. The exam consists of 4 questions for a total of 90 points. There 10 points are given for writing your own answers and not cheating. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) (25 points) Solve

$$\frac{d^2y}{dt^2} = 16\frac{d^2y}{dx^2} \quad 0 < x < 2 \quad t > 0$$

subject to the conditions $y(0,t) = 0$, $y(2,t) = 0$, $y(x,0) = 6\sin(\pi x) - 2\sin(4\pi x)$, $y_t(x,0) = 0$ and $|y(x,t)| < M$.

(2) (15 points total, 5 points each) Decide whether or not the following is true or false and provide justification for your answer:

a) The sequence

$$f_n(x) = \frac{1}{\pi}\tan^{-1}(nx) + \frac{1}{2}$$

converges uniformly

b) If $f(x)$ is a piecewise continuous function, its absolute value $|f(x)|$ is also piecewise continuous.

c) if $f(x)$ is piecewise C^1 function, its Fourier series always converges to a piecewise C^1 function.

(3) Let Ω be an open, bounded domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$

a)(20 points) Define the Green's function $G(x,y)$ for the Dirichlet problem

$$-\Delta u = f(x) \quad x \in \Omega$$
$$u = g(x) \quad x \in \partial\Omega$$

Give a proof that G is unique, that $G(x,y) = G(y,x)$ and that

$$u(x) = \int_{\Omega} G(x,y)f(y)\,dy + \int_{\partial\Omega} G(x,y)g(y)\,dy$$

is a solution of this boundary value problem. (Hint, use Green's second identity)

b) (5 points) Suppose that $d = 1$ and $\Omega = (0,\pi)$. Show that

$$G(x,\xi) = \begin{cases} \frac{x(\pi-\xi)}{\pi} & x < \xi \\ \frac{\xi(\pi-x)}{\pi} & x > \xi \end{cases}$$

(4) a)(20 points) Use the Fourier transform to find the solution to

$$\partial_t u = \Delta u$$
$$u(0,x) = e^{-x^2}$$

for $u(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. You may use without proof the fact that

$$\int_{-\infty}^{\infty} e^{-cy^2} e^{iy\eta}\,dy = \sqrt{\frac{\pi}{c}}e^{-\frac{\eta^2}{4c}} \tag{0.1}$$

for some constant $c > 0$.

b)(5 points) what happens to the solution as $t \to \infty$? (justify your answer)